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Note

Exact Triple Integrals of Beam Functions

In solving certain fluid dynamic problems it is often convenient to expand the prevailing velocity fields in a complete set of orthogonal functions which together with their first derivatives vanish at the ends of the chosen interval [1]. These functions are constructed from the eigenvalue problem defined by the equation

$$\frac{d^4y}{dx^4} = \alpha^4 y \tag{1}$$

and the boundary conditions

$$y = \frac{dy}{dx} = 0 \qquad \text{at} \qquad x = \pm\frac{1}{2}. \tag{2}$$

They fall into two non-combining groups of even functions $C_m(x)$ and odd functions $S_m(x)$ which take the forms

$$C_m(x) = \frac{\cosh \lambda_m x}{\cosh \frac{1}{2}\lambda_m} - \frac{\cos \lambda_m x}{\cos \frac{1}{2}\lambda_m} \tag{3}$$

and

$$S_m(x) = \frac{\sinh \mu_m x}{\sinh \frac{1}{2}\mu_m} - \frac{\sin \mu_m x}{\sin \frac{1}{2}\mu_m}. \tag{4}$$

Here λ_m and μ_m are the roots of the characteristic equations

$$\tanh \tfrac{1}{2}\lambda + \tan \tfrac{1}{2}\lambda = 0 \tag{5}$$

and

$$\coth \tfrac{1}{2}\mu - \cot \tfrac{1}{2}\mu = 0, \tag{6}$$

respectively.

These functions were first discussed by Rayleigh [2] in the theory of vibrating beams and are also referred to as beam functions. In the application of these functions to linear problems, one needs to evaluate definite double integrals involving beam functions and their derivatives. These integrals can be evaluated quite simply by using the fact that the beam functions satisfy the differential equation (1) and by carrying out four partial integrations [3]. However, in the application of these functions to

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non-linear problems one encounters definite triple integrals involving them [4, 5]. Usually these integrals are evaluated numerically using Gaussian quadrature formulae.

We encountered these triple integrals in the application of the Galerkin method to the problem of heat and mass transfer across rectangular enclosures. Instead of evaluating them numerically, we extended the technique described by Reid and Harris [3] to obtain the exact solution of these triple integrals. In this process we obtained four linear simultaneous equations with triple integrals as unknowns. These equations were solved exactly to obtain the closed form solution. Such closed form representations have been shown to be useful in solving nonlinear hydrodynamic problems by series expansion[4, 6, 7]. Hence, we present below these integrals in a general form.

We use the following representation for the beam functions and the associated triple integrals

$$B_m(1, x) = C_m(x), \tag{7}$$

$$B_m(2, x) = S_m(x), \tag{8}$$

$$v_m(1) = \lambda_m^4, \tag{9}$$

$$v_m(2) = \mu_m^4, \tag{10}$$

$$I_{mpa}(ijk) = \int_{-1/2}^{1/2} B_m(i, x)\, B_p'(j, x)\, B_a(k, x), \tag{11}$$

$$J_{mpa}(ijk) = \int_{-1/2}^{1/2} B_m(i, x)\, B_p'''(j, x)\, B_a(k, x), \tag{12}$$

$$K_{mpa}(ijk) = \int_{-1/2}^{1/2} B_m'(i, x)\, B_p''(j, x)\, B_a(k, x), \tag{13}$$

where i, j, k take the values 1 or 2. Then the exact integrals are:

$$I_{mpa}(ijk) = \frac{1}{D}\, [4(bz + 4cv_p(j))\, T_2 + 2(2c - 3b)\, T_1], \tag{14}$$

$$J_{mpa}(ijk) = \frac{2dv_p(j)}{D}\, [2(z + 4v_m(i))\, T_2 - T_1], \tag{15}$$

$$K_{mpa}(ijk) = \frac{d}{D}\, [-8v_m(i)\, v_p(j)\, T_2 + (z + 4v_p(j))\, T_1], \tag{16}$$

where

$$\begin{aligned} c = {} & 2B_m'''(i, \tfrac{1}{2})\, B_p''(j, \tfrac{1}{2})\, B_a'''(k, \tfrac{1}{2}) \\ & - 2B_m'''(i, \tfrac{1}{2})\, B_p'''(j, \tfrac{1}{2})\, B_a''(k, \tfrac{1}{2}), \end{aligned} \tag{17}$$

$$b = 2B''_m(i, \tfrac{1}{2})\, B'''_p(j, \tfrac{1}{2})\, B'''_a(k, \tfrac{1}{2}) - 2B'''_m(i, \tfrac{1}{2})\, B'''_p(j, \tfrac{1}{2})\, B''_a(k, \tfrac{1}{2}), \tag{18}$$

$$d = -2B''_m(i, \tfrac{1}{2})\, B''_p(j, \tfrac{1}{2})\, B''_a(k, \tfrac{1}{2}) \tag{19}$$

$$z = v_a(k) - v_m(i) - v_p(j), \tag{20}$$

$$T_1 = z^2 + 4v_m(i)\, v_p(j), \tag{21}$$

$$T_2 = 3z + 4v_m(i) + 4v_p(j), \tag{22}$$

$$D = T_1^2 - 16v_m(i)\, v_p(j)\, T_2^2. \tag{23}$$

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